Review of "Where Mathematics comes from: How the Embodied Mind Brings Mathematics Into Being" By George Lakoff and Rafael E. Nunez

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ABSTRACT
In this paper we will review a multidisciplinary book entitled as "where mathematics comes from: how the embodied mind brings mathematics into being". Our review is divided into two parts: the first part is a summary of the book with focusing on its most important ideas which are embodiment of basic arithmetic, metaphorizing capacity, and philosophy of embodied mathematics, and the second part is our comments about the book, its structure and research methodology.

Keywords
Cognitive science, Mathematical idea analysis, Embodiment, Philosophy, Mathematics, Linguistics, Neuroscience.

Categories and Subject Descriptors
Human Factors, Languages, Embodiment, Cognitive Psychology.

INTRODUCTION
This book is a pioneering research that attempted to understand the evolution of mathematics, theorems, and proofs, and how we as humans perceive these information. It is written by George Lakoff and Rafael Nunez. Lakoff is a cognitive science and linguistics professor. He is most famous for his ideas about the centrality of metaphor to human thinking, political behavior and society. For Lakoff the greater the level of abstraction the more layers of metaphor are required to express it. The authors investigated the relationship between cognitive science, neurology, embodiment, linguistics, and mathematics though their multidisciplinary research. The book is divided into five parts. The first part is an introduction that spans innate arithmetic, basic results in cognitive science, basic metaphors grounding our understanding of arithmetic, and the question of where the laws of arithmetic come from. The second part is about grounding and conceptualization of sets, logic, and forms of abstract algebra such as groups. In part three the authors discussed the concept of infinity, which is a fundamental concept in mathematics and extended to points of infinity, infinite sets, infinite decimals, limits, and mathematical induction. Then the authors point out in part four the implications of this type of analysis for an understanding of the continuum and for continuity and the real numbers. The fifth part is the conclusion of the book; the authors discuss their hypothesis of mathematics embodiment and its philosophy. Finally, they illustrated the power of their approach the mathematical idea analysis through an extensive case study that combines analytic geometry and trigonometry, exponentials and logarithms, and imaginary numbers.

EMBODIED ARITHMETIC
The main goal of this book is to answer the question appeared at its title: where mathematics comes from? And

as also revealed from the book title, the authors find their answer in embodiment. They claim that the mathematics we used to describe as disembodied is in fact embodied. Humans use their bodies, mind, and brain to both form and understand mathematics. All mathematical content resides in embodied ideas and many of the most basic, as well as the most sophisticated, mathematical ideas are metaphorical.

The authors take intuition as their starting point. They began their first chapter by surveying various experiments held by cognitive scientists that prove that humans (and even some animals) are born with a capacity for subitizing very small numbers of objects and doing the arithmetic (addition and subtraction) of small numbers. Subitizing is to recognize the number of up to four objects quickly and accurately. It is a vital visual perception skill that is a precursor to basic math skills including numeracy and visual counting. The authors further emphasize their point by listing references to neurological studies. These studies illustrate that specific parts of the brain are responsible for specific arithmetic capabilities. For example, the capacity for basic arithmetic is separate from the capacity for rote memorization of addition and multiplication tables. Algebraic abilities are also localized separately from basic arithmetic.

GROUNDING METAPHORS

The question arises then is how humans used their built-in arithmetic capacities to develop more sophisticated mathematics such as algebra, trigonometry, and calculus, etc. Lakoff and Nunez gave one compact answer to this tough question. Their thesis is that all mathematical ideas arise as metaphors where mathematical ideas are ways of mathematicizing ordinary ideas. They found that there are two types of conceptual metaphor used in projecting from subitizing, counting, and the simplest arithmetic of newborns to arithmetic of natural numbers. The first is grounding metaphor which allows projections from everyday experience (like putting things into piles) onto abstract concepts (like addition), and the second is linking metaphor that yields sophisticated abstract ideas and make connections between different branches of mathematics.

It is basic to the authors' arguments that the notions in the left-hand column have literal meaning, while the notions in the right-hand column do not. The notions in the right-hand column gain their meanings from the notions in the left-hand column via the metaphor. Each conceptual metaphor has entailments, which for this metaphor the authors describe as follows:

Take the basic truths about collections of physical objects. Map them onto statements about numbers, using the metaphorical mapping. The result is a set of 'truths' about the natural numbers under the operations of addition and subtraction.

Lakoff and Nunez claimed that abstract concepts are always rooted, through combination of linking and grounding metaphors, to sensory-motor experiences. The authors' notion of cognitive metaphor can be illustrated by the "Arithmetic Is Object Collection" metaphor. This metaphor, as with all cognitive metaphors, consists of two domains and a mapping:

- Source domain: collections of objects (based on our commonest experiences with grouping objects).
- Target domain: arithmetic (natural numbers with addition and subtraction).
- Mapping across the domains as described in the following table:

Arithmetic Is Object Collection Metaphor		
Source Domain		*Target Domain*
Object Collection		Arithmetic
Collections of objects of the same size	→	Natural numbers
The size of the collection	→	The size of the number
Bigger	→	Greater
Smaller	→	Less
The smallest collection	→	The unit (One)
Putting collections together	→	Addition
Taking a smaller collection from a larger collection	→	Subtraction

Table 1. Arithmetic is object collection metaphor.

This metaphor connects experiences with collections of objects on the one hand and the basic arithmetic operations on the other. For example, joining two collections of objects is correspondent to adding two numbers while taking a smaller collection from a larger collection is correspondent to subtraction. The commutative law of addition corresponds to the fact that when two collections are thrown together, it does not matter which goes first.

The authors mention other grounding metaphors to explain other arithmetic operations such as "Arithmetic as Object Construction", and "The Measuring Stick Metaphor". Arithmetic operations such as adding positive and negative numbers may be understood through "Motion Along a Path" metaphor by referring to forward and backward trips along a linear path. They called these grounding metaphors the 4Gs. The significance of the 4Gs is that they allow humans who have an innate capacity to form metaphors, to extend arithmetic beyond the small amount that we are born with.

The authors then repeated their analysis to other mathematical concepts in the following chapters. They turned to the grounding and conceptualization of sets, logic, and forms of abstract algebra such as groups. From the fundamental mathematical ideas that are discovered to be inherently metaphorical are:

- Boole's algebra of classes: where the formation of classes of objects is conceptualized metaphorically in terms of algebraic operations and elements: plus, times, zero, one, and so on.
- Symbolic logic: where reasoning is conceptualized metaphorically as mathematical calculation using symbols.
- Trigonometric functions: where angles are conceptualized metaphorically as numbers.
- The complex plane: where multiplication is conceptualized metaphorically in terms of rotation.

INFINITY CONCEPTUALIZATION

Moving to chapter 8, Lakoff and Nunez try to answer a challenging question: how the concept of infinity can be embodied although every thing related to humans is finite, our bodies and our experiences are finite. They find their answer in literature of linguistics, which is the aspectual system. It characterizes the structure of event-concepts. They said that aspectual system is embodied in the motor control system of the brain. They claimed that a process is infinite if it continues/iterates indefinitely without stopping such as breathing and that is the literal concept of infinity.

All the mathematical uses of infinity are supposed to arise from a single general conceptual metaphor in which processes that go on indefinitely are conceptualized as having an end and an ultimate result. This metaphor is "The Basic Metaphor of Infinity" (BMI) which is a general cognitive mechanism. Its details are as follows:

- Source domain: completed iterative processes
- Target domain: iterative processes that go on and on
- Mapping across the domains as described in the following table:

The Basic Metaphor of Infinity		
Source Domain		*Target Domain*
Completed Iterative Processes		Iterative Processes That Go On and On
The beginning state	→	The beginning state
State resulting from the initial stage of the process	→	State resulting from the initial stage of the process
The process: From a given intermediate state, produce the next state	→	The process: From a given intermediate state, produce the next state
The intermediate result after that iteration of the process	→	The intermediate result after that iteration of the process
The final resultant state	→	**"The final resultant state" (actual infinity)**
Entailment E: The final resultant state is unique and follows every nonfinal state	→	**Entailment E: The final resultant state is unique and follows every nonfinal state**

Table 2. The basic metaphor of infinity.

BMI is a mapping from the source domain "the general idea of an iterative process that reaches a completion" such as walking to a destination to the target domain "any iterative process that potentially goes on and on" like counting 1,2,3... A completed iterative process has 4 parts, all literal: the beginning state, the process that from an intermediate state produces the next state, an intermediate state, and the final resultant state that is unique and follows every non-final state. These are mapped onto 4 parts (with the same names and descriptions) of an Iterative Process That Goes On and On, where the first three parts have literal meaning but the last part (the 'final resultant state') has meaning only metaphorically from the cognitive mapping. The authors' explanation for this metaphor is that humans think about infinite processes through their knowledge of finite ones. They believed that BMI metaphor can explain most if not all infinite notions in mathematics and that all cases of actual infinity are special cases of" a single cognitive metaphor which is the Basic Metaphor of Infinity.

LINKING METAPHORS

After seeing various examples of grounding metaphors, the authors move to more sophisticated mathematical concepts that can not be conceptualized by the means of grounding metaphors. Much of the abstraction of higher mathematics is a consequence of the systematic layering of metaphor upon metaphor, often over the course of centuries. Each metaphorical layer carries inferential structure systematically from source domains to target domains. This systematic structure needs to be revealed by detailed metaphorical analysis.

Linking metaphors are needed to explain sophisticated mathematical concepts. They occur whenever one branch of mathematics is used to model another. They are central to the creation of new mathematical concepts and new branches of mathematics such as analytic geometry and trigonometry.

An example of linking metaphor is “Numbers Are Points on a Line” metaphor. This metaphor constitutes our nontechnical understanding of numbers as points on a line as follows:

- Source domain: points on a line
- Target domain: a collection of numbers
- Mapping across the domains as described in the following table:

Numbers Are Points on a Line

Source Domain Points on a Line		*Target Domain* A Collection of Numbers
A point P on a line	→	A number P’
A Point O	→	Zero
A point I to the right of O	→	One
Point P is to the right of point Q	→	Number P’ is greater than number Q’
Point Q is to the left of point P	→	Number Q’ is less than number P’
Point P is in the same location as point Q	→	Number P’ equals number Q’
Points to the left of O	→	Negative numbers
The distance between O and P	→	The absolute value of number P’

Table 3. Numbers are points on a line metaphor.

Unlike grounding metaphors, “Numbers Are Points on a Line ” metaphor maps source domain to target domain where both of them are mathematical sources. This metaphor is to explain what a number line is, as visualized in graphs for example. It allows us to conceptualize one mathematical domain in terms of another mathematical domain.

After going through different kinds of metaphors mentioned by the authors to explain mathematical ideas, we can sum up their main idea as follows: you *understand* a piece of mathematics if you can do the following:

- Explain mathematical concepts and facts in terms of simpler concepts and facts.
- Easily make logical connections between different facts and concepts.
- Recognize the connection when you encounter something new (inside or outside of mathematics) that's close to the mathematics you understand.
- Identify the principles in the given piece of mathematics that make everything work. (i.e., you can see past the clutter.)

THE THEORY AND PHILOSOPHY OF EMBODIED MATHEMATICS

Chapters 15 and 16 treated the philosophical part of the book. The authors describe mathematics as a system of human concepts that makes extraordinary use of the ordinary tools of human cognition. It is special in that it is stable across time and communities, precise, generalizable, symbolizable, calculable, consistent within each of its subject matters, universally available, and effective for precisely conceptualizing a large number of aspects of the world as we expect it. The authors critique a mythology they named it “the romance of mathematics”, they ran up against describing mathematics as:

- Mathematics is abstract and disembodied.
- Human mathematics is a part of abstract mathematics.
- Mathematics is part of the physical universe.

- Mathematics characterizes logic and structures reason.
- Mathematical truth is universal, absolute, and certain.

The authors challenged these descriptions of mathematics and they argue that mathematics is embodied. They based their argument on recent research in neuroscience, cognitive science, and the history of mathematics. Their hypothesis about embodied mathematics makes the following claims:

- Mathematics is a product of humans, so it is limited by the nature of our brains, bodies, conceptual systems, and cultures.
- Humans can subitize, which is clearly an embodied capacity.
- The subject matters of mathematics arise from human concerns and activities.
- The mathematical aspect of these concerns is precision.
- Precision is greatly enhanced by the human capacity to symbolize.
- Conceptual metaphor is a neurally embodied fundamental cognitive mechanism that allows us to use the inferential structure of one domain to reason about another.
- Mathematical inferences tend not to change over time or space or culture once it is approved firmly within a community of mathematicians.

These properties established the theory of embodied mathematics.

The authors summarize their view of the philosophy of mathematics with the statement: "Mathematics as we know it is human mathematics, a product of the human mind. Where does mathematics come from? It comes from us! We create it, but it is not arbitrary-not a mere historically contingent social construction. What makes mathematics non-arbitrary is that it uses the basic conceptual mechanisms of the embodied human mind as it has evolved in the real world. Mathematics is a product of the neural capacities of our brains, the nature of our bodies, our evolution, our environment, and our long social and cultural history. "

At the end of Chapter 16 the authors briefed their philosophical view of mathematics in one poetic statement ***"The portrait of mathematics has a human face".***

CASE STUDY

The last part of the book is a case study, the authors tried to apply their mathematical idea analysis technique to Euler equation $e^{\pi i} + 1 = 0$. The authors said that they want to characterize the meaning of the equation and provide an understanding of it. In order to explain this equation, they looked at the conceptual metaphors underlying analytic geometry and trigonometry, exponentials and logarithms, imaginary numbers, and the cognitive mechanisms that combine all of the previous. They found that that the significance of $e^{\pi i} + 1 = 0$ is a conceptual significance. The numbers e, i,π, 1, and 0 are not just numbers like any other numbers, these numbers have conceptual meanings in a system of common, important nonmathematical concepts, like change, acceleration, recurrence, rotation, and self-regulation. They are not mere numbers; they are the arithmetizations of concepts. It is no accident that our branches of mathematics are linked in the way they are. The way the branches of mathematics are interrelated is a consequence of what is important to us in our daily lives and how we conceptualize those concerns.

This case study is a detailed illustration of how the cognitive mechanisms discussed through the book can explain classical mathematics. Moreover, this case study shows the power of the authors' developed technique, "mathematical idea analysis", as it shows how just one equation can bring a various collection of ideas together although the equation consists of only numbers. The purpose of the mathematical idea analysis is to provide a new level of understanding in mathematics. It seeks to explain why theorems are true on the basis of what they mean. It may require some complicated analysis:

- Tracing through a complex mathematical idea network to see what the ultimate grounding metaphors in the network are.
- Isolating the linking metaphors to see how basic grounded ideas are linked together.
- Figuring out how the immediate understanding provided by the individual grounding metaphors permits one to comprehend the complex idea as a whole.

BOOK SUMMARY

The book introduces to us a novel multidisciplinary research that is important for both mathematicians and cognition scientists. It fuses psychology, philosophy, and mathematics together. The book motivates research in cognitive science of mathematics that is stimulating yet challenging.

The importance of this book can be summarized in the following points:

- Lakoff and Nunez grounded a new field of research: cognitive science of mathematics.
- They present a new theoretical framework for understanding the human nature of Mathematics and its foundations.
- The authors developed the "Mathematical idea analysis" technique that they used to uncover metaphorical elements in mathematical ideas.
- The book provides mathematics teachers with new tools and devices to use in their teaching activities.

BOOK REVIEW

In this section, we will write our comments and notes on the book and how this pioneering research can be enriched and strengthened.

1. George Lakoff is a linguistics professor and Rafael Nunez is a psychology researcher. They succeeded to analyze mathematical concepts from their perspectives depending on their different scientific backgrounds without going to deep details of mathematics. This can be considered as a privilege for the general reader who does not have strong mathematical background. On the other hand, if a professional mathematician shared in this research and lay down more deep details and analysis, the book could be more useful for the mathematics research community.

2. The authors limited their view of embodiment and its role in developing mathematics in humans' innate arithmetic and their brain parts responsible for various mathematical functions. However embodiment can be further extended in its effect on the evolution of mathematics. For example, humans chose to express all numbers using ten digits. Is that because they used their ten fingers in counting? If people were normally born with six fingers in every hand, would that make any difference in our numeric system? Lakoff and Neon focused on the cognition and neurological aspects of embodiment. They did not refer to the influence of our bodies' shape on our innate arithmetic.

3. The authors focused their entire thesis on the idea of metaphors. They tried to explain all mathematical ideas through various metaphors. In fact I do not agree with them in some of their analysis. While some mathematical ideas can be explained via metaphors, others may not. For example, they discussed the metaphorical meaning of "Zero" by projecting it into the four grounding metaphors: object collection, object construction, measuring stick, and motion along a line. They said that Zero according to these four metaphors can symbolically denote emptiness, nothingness, lack, absence, and destruction. It appears that Lakoff and Neon are trying to explain the meaning of Zero instead of investigating where it comes from. Invention of zero was not that straight forward in the history of mathematics. Mathematicians spend centuries asking themselves how nothing can be something. Another critical problem with their metaphorical meaning of zero is that they make "Zero" and "Phi" interchangeable. Of course zero and phi are two different things in literature of mathematics and they mean two different things. Unfortunately the authors did not differentiate them appropriately.

4. While reading this book a question enforces itself arose: to what extent did the scientific background of the authors biased their research? Most of the metaphors used at this book come from linguists' perspective. Most probably that other metaphors can be more convenient than the mentioned ones in the book.

5. Lakoff and Nunez hypothesize that all people understand mathematics using the same metaphors. They did not question the influence of any factors that could affect the process of understanding mathematics. Such factors can be cultural differences, age (naturally will affect level of experience), and personal differences (including educational level).

6. Although the author's main goal is to try finding out where mathematics comes from, they did not interview any mathematician or did any kind of research to know how mathematicians work, develop, and explain mathematics. I think it will be really useful taking mathematicians' experience for better understanding how mathematics is embodied.

7. The second objective of the book is to figure out how people think about mathematics and learn its rules. In spite this goal; the authors did not mention any research about mathematical education from the

perspective of mathematics teachers or students. Their metaphors are completely dependent on their hypothesis that relies on linguistics to great extent.

8. The structure of the book goes as follows, the first two chapters are interesting introduction to cognitive science and how it can be applied to mathematics. The last two chapters are the conclusion of the book, and describing the new philosophy of mathematics and showing how mathematics is embodied. The chapters in between (14 chapters) are repetitive examples of metaphorical hypothesis for various mathematical concepts. From my point of view the most important chapters are the first and the last two chapters. The rest of the book can be reduced as the examples given to prove the authors' hypothesis are repeating themselves. I would prefer that the authors dedicate more chapters for the theory, analysis, and discussion rather than their excessive care with examples. Also the authors were not fair in choosing their examples as most of their work focused on algebra, calculus, and infinity, and they barely touched geometry in their metaphors.

9. The authors chose to list their references as various categories depending on the discipline that the reference belongs to. So they ended up with six lists of references, each of them is sorted alphabetically according to author's name. I find this style of writing references is hard to follow.

CONCLUSION

To sum up this review, "Where Mathematics Comes From" is a "think out of the box" book. The authors had the courage to question mathematical believes that lasts for centuries and alter our way of looking at mathematics. They introduced new research area and further research is welcomed and even needed to form more accurate conception of our embodied mathematics. Mathematics will not be the same, you used to, after reading this book.

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